Preparation Uncertainty Relations and Weak Values

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October 20, 2016

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- To this weird list, the weak value adds another twist making quantum theory even more stranger than before.

- The concept of weak value was introduced by Aharonov, Albert and Vaidman while investigating the properties of a quantum system in pre and post-selected ensembles. If a system is weakly coupled to an apparatus, then upon post-selection of the system state, the apparatus pointer observable is shifted by (the real part of) a weak value

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- Obtain a strong uncertainty relation for unitary operators– a new preparation UR.

- Obtain a complementarity relation for the weak values of two non-commuting projection operators, that restricts the degree to which they can take anomalous values outside the interval $[0, 1]$.

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- If we weakly measure the projection $|a\rangle\langle a|$ with postselection on projection $|b\rangle\langle b|$, and vice versa then the product of the corresponding weak values for these complementary scenarios is restricted to be a positive real number, no greater than $|\langle a|b\rangle|^2$, with equality for all pure initial states.

M. J. W. Hall, A. K. Pati and J. Wu, Phys. Rev. A **93**, 052118 (2016)

Weak Measurement

- Prepare system in a quantum state (pre-selection).

- Weakly couple system and apparatus.

- Perform strong measurement on the system (post-selection).

- Apparatus state is affected by the weak value.

Weak Measurement

With pre and post-selection

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- Weak measurement idea: Aharonov-Albert-Vaidman [1]

- Interaction Hamiltonian $H_{\mathrm{SA}} = g\delta(t - T)O \otimes P$.

[1] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).

Weak Measurement

With pre and post-selection

- Weak measurement idea: Aharonov-Albert-Vaidman [1]
- Interaction Hamiltonian $H_{\rm SA} = g\delta(t - T) O \otimes P$.
- System and apparatus evolve

$$|\psi_i\rangle \otimes |\phi\rangle \to \exp(-igO \otimes P)|\psi_i\rangle \otimes |\phi\rangle$$

$$\approx (1 - igO \otimes P)|\psi_i\rangle \otimes |\phi\rangle$$

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The Weak Measurement

The Weak Measurement

- Upon post-selection in $|\psi_f\rangle$, final apparatus state

$$|\phi_f\rangle = (1 - g\frac{\langle\psi_f|O|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}P)|\phi\rangle$$
$$= \exp(-ig\langle O\rangle_w P)|\phi\rangle$$

- Weak value

$$\langle O\rangle_w = \frac{\langle\psi_f|O|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \quad (1)$$

- Weak value can be a complex number and can take large value.

- Strong measurement reveals one aspect of quantum state. Weak measurement can reveal incompatible aspects of quantum state.
- Measure wave function directly [Lundeen *et al* NATURE (2011)].
- Amplify weak signals.
- Probing average trajectory in interference setup.
- Weak values helps to measure average of any non-Hermitian operators in a quantum state [Pati-Singh-Sinha, PRA (2015)].

Average of weak values and observables

Average of weak values and observables

- Weak value of a Hermitian operator A

$$A_w(\phi|\psi) := \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \tag{2}$$

- Postselected state $|\phi\rangle$ may be replaced by $|m\rangle$ with $M \equiv \{|m\rangle\langle m|\}$ and $\sum_m |m\rangle\langle m| = \hat{1}$, leading to the expression

$$A_w(m|\psi) := \frac{\langle m|A|\psi\rangle}{\langle m|\psi\rangle}, \tag{3}$$

- Since the probability of measurement outcome m on state $|\psi\rangle$ is $p(m|\psi) = |\langle m|\psi\rangle|^2$, the average of the weak value

$$\langle A_w\rangle_p := \sum_m p(m|\psi)\, A_w(m|\psi) = \langle\psi|A|\psi\rangle =: \langle A\rangle_\psi, \tag{4}$$

  This reconstruction formula holds for non-Hermitian operators A via linearity.

- Remarkably, the above reconstruction formula for the average of an observable may be extended to a similar formula for products [Hosoya-Shikano] (generalised here to arbitrary non-Hermitian operators):

$$\sum_m p(m|\psi)\, A_w(m|\psi)^*\, B_w(m|\psi) = \langle\psi|A^\dagger B|\psi\rangle$$

$$\langle A_w^* B_w\rangle_p = \langle A^\dagger B\rangle_\psi, \tag{5}$$

- Average of an operator product, with respect to a quantum state $|\psi\rangle$, can be replaced by a average of a product of weak values, with respect to the classical probability distribution $p(m|\psi)$.

- The product representation formula has a clear operational significance. For example, if the weak values of two Hermitian operators A and B, postselected on measurement M, are determined experimentally, then one can immediately recover not only the averages of the observables A and B, but also the averages of A^2, B^2, and AB—and, hence, the variances and covariances of A and B.

- One can also experimentally recover the average of the operator $(A - B)^2$ from the weak values of A and B, where this average appears in various error-disturbance and joint-measurement uncertainty relations.

- Weak values also provide a (complex) hidden variable model for the averages of a given set of quantum observables and their pairwise products.
- For example, the average values of all linear and quadratic functions of the annihilation and creation operators a and $a^\dagger$ of a single mode field, including the quadrature observable $X_\theta = ae^{i\theta} + a^\dagger e^{-i\theta}$ and the number operator $a^\dagger a$, can be modelled for any state $|\psi\rangle$, via the corresponding weak values $a_w(m|\psi)$ and $a^\dagger_w(m|\psi)$ and classical probability distribution $p(m|\psi)$ (and for any choice of POVM $M \equiv \{|m\rangle\langle m|\}$).

Uncertainty relations from weak values

Uncertainty relations from weak values

- Complex random variables are standard tools in classical signal processing and information theory.
- A complex random variable $\alpha = \alpha_1 + i\alpha_2$ is described by some real and positive probability density $p(\alpha)$. The expectation value of function $f(\alpha)$ is then given by $\langle f(\alpha) \rangle := \int d\alpha \, p(\alpha) f(\alpha)$, where the integral is over the complex plane with respect to the uniform measure.

- The variance of α is just the average mean square distance between α and its mean value, i.e.,

$$Var\,\alpha := \langle\, |\alpha - \langle\alpha\rangle|^2 \rangle = \langle\, |\alpha|^2 \rangle - |\langle\alpha\rangle|^2. \tag{6}$$

- Similarly, the covariance of two such random variables, α and β, with respect to a joint probability distribution $p(\alpha, \beta)$, is defined by

$$Cov(\alpha, \beta) := \langle\, (\alpha - \langle\alpha\rangle)^*(\beta - \langle\beta\rangle)\rangle = \langle\, \alpha^*\beta\rangle - \langle\alpha^*\rangle\langle\beta\rangle \tag{7}$$

with $\langle f(\alpha, \beta)\rangle := \int d\alpha d\beta\, p(\alpha, \beta)\, f(\alpha, \beta)$. Thus, $Var\,\alpha = Cov(\alpha, \alpha)$, and one immediately has the *classical* uncertainty relation

UR for complex random variable

UR for complex random variable

- $$Var\,\alpha\ Var\,\beta \geq |Cov(\alpha,\beta)|^2 \quad (8)$$
  from the Schwarz inequality for complex numbers.
- Choosing $\alpha = A_w(m|\psi)$, $\beta = B_w(m|\psi)$, and probability distribution $p(m|\psi)$, this classical uncertainty relation reduces to
  $$Var_p A_w\ Var_p B_w \geq |Cov_p(A_w, B_w)|^2 . \quad (9)$$
  for the case of weak values.

Heisenberg inequality as a classical uncertainty relation

Heisenberg inequality as a classical uncertainty relation

- Heisenberg UR for two non-commuting observables follows directly from the representation formula and the classical UR.
- For two Hermitian operators A and B, substituting the first equation into the second and decomposing the right hand side into real and imaginary parts yields

$$Var_\psi A \, Var_\psi B \geq |\langle AB\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi|^2$$

$$= Cov_\psi(A,B)^2 + \frac{1}{4} |\langle [A,B]\rangle_\psi|^2 \,,$$

  with the quantum covariance defined by $Cov_\psi(A,B) := \frac{1}{2}\langle AB + BA\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi$.
- Thus, the standard quantum uncertainty relation may be reinterpreted as a *classical* uncertainty relation for weak values.

- It is a curious fact that one of the fundamental relations of quantum mechanics, namely, the Heisenberg uncertainty relation, can be understood as a classical uncertainty relation for complex random variables.
- In this sense the weak value approach classicalizes the UR.
- Recently, stronger uncertainty relations have been proved which go beyond the Robertson-Schrödinger uncertainty relation [Maccone-Pati, PRL (2014)]. It may be worth to explore if one can view these also as classical uncertainty relations for weak values.

Uncertainty relations for non-Hermitian operators

Uncertainty relations for non-Hermitian operators

- Variance of a general operator [Levy-Leblond (1976), Anandan (1990), Pati-Singh-Sinha (2015)]

$$Var_{\psi}A := \langle A^{\dagger}A\rangle_{\psi} - |\langle A\rangle_{\psi}|^2 = Var_p A_w, \tag{10}$$

  where the second equality follows from the product representation formula.
- This has the desirable properties of vanishing if and only if $|\psi\rangle$ is an eigenstate of A, and of reducing to the usual variance in the Hermitian case $A = A^{\dagger}$.
- Annihilation and creation operators a and $a^{\dagger}$ of a single-mode bosonic field satisfy $[a, a^{\dagger}] = 1$, their variances are related by

$$Var\, a^{\dagger} = \langle aa^{\dagger}\rangle - |\langle a\rangle|^2 = Var\, a + 1 \geq 1, \tag{11}$$

  implying immediately that $a^{\dagger}$ has no eigenstates.

Uncertainty relations ....

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- The classical uncertainty relation yields the generalization

$$Var_\psi A \, Var_\psi B \geq |Cov_p(A_w, B_w)|^2$$

$$= |\langle A^\dagger B\rangle_\psi - \langle A^\dagger\rangle_\psi \langle B\rangle_\psi|^2$$

- UR for general operators A and B (second line follows via the product representation formula and the definition of covariance).
- This uncertainty relation was obtained by [Pati-Singh-Sinha PRA (2015)]. This is equivalent to a 'classical' uncertainty relation for complx weak values.

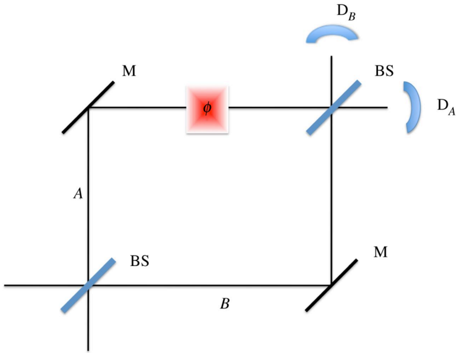
- Given a quantum system 'prepared' in a state $|\psi\rangle$ it can change either (i) by unitary transformation, (ii) by measurement process or (iii) by general quantum operation.
- Preparation URs based on Hermitian observables need measuements of two non-commuting observables on identically prepared quantum system.
- Question: Is it possible to test preparation uncertainty without doing measurement of two non-commuting observables?
- YES!
- You need UR relation for two unitary operators.

Uncertainty relations for unitary operators

Uncertainty relations for unitary operators

- Uncertainty in unitary operator U: $\Delta U^2 = 1 - |\langle\psi|U|\psi\rangle|^2$
- Uncertainty in unitary operator V: $\Delta U^2 = 1 - |\langle\psi|U|\psi\rangle|^2$
- If we send a particle in Mach-Zehnder interferometer and apply U in one arm of the interferometer, visibility in interference $|\langle\psi|U|\psi\rangle|$.
- Uncertainty in U places restriction on the visibility.

Mach-Zehnder Interferometer



• Uncertainty relation for two unitary operators shows that quantum states cannot be prepared such that the sum of interference visibilities due to two non-commuting unitary operators will be non-trivially upper bounded.

• Unecratinty relation for two unitary operators [Bagchi-Pati, PRA (2016)]

$$\Delta U^2 + \Delta V^2 \geq 1 + |\langle\psi_U|\psi_V\rangle|^2 - 2\cos\phi|\Delta^{(3)}|, \quad (12)$$

where

$$|\langle\psi_U|\psi_V\rangle|^2 = \langle\psi|V^\dagger U|\psi\rangle\langle\psi|U^\dagger V|\psi\rangle$$

$$\Delta^{(3)} = \langle\psi|\psi_U\rangle\langle\psi_U|\psi_V\rangle\langle\psi_V|\psi\rangle$$

is the 3-point Bargmann invariant

- As an application of the UR [Hall-Pati-Wu] consider the case of two unitary operators U and V. Define

$$u := |\langle U \rangle_\psi|, \qquad v := |\langle V \rangle_\psi|,$$

- Uncertainty Relation for two Unitaries is equivalent to [Hall-Pati-Wu, PRA (2016)]

$$u^2 + v^2 - 2uv|\langle U^\dagger V \rangle_\psi| \cos \Phi \leq 1 - |\langle U^\dagger V \rangle_\psi|^2. \qquad (13)$$

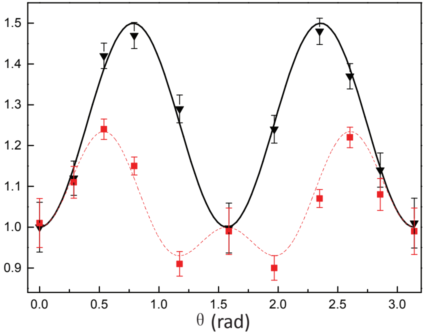
- Note that $|\langle U^\dagger V \rangle_\psi|^2$ is the overlap of the states $U|\psi\rangle$ and $V|\psi\rangle$. Hence, the overlap plays a role analogous to the commutator in the Heisenberg uncertainty relation.

• Application: Abbott-Alzieu-Hall-Branciard tight state-independent uncertainty relations for qubits [Mathematics 4(1), 8, (2016)] can be obtained from the UR for unitary perators simply by letting unitary operators to be the Pauli observables, i.e., $U = a.\sigma$ and $V = b.\sigma$ [Bagchi-Pati, PRA (2016)].

$$\Delta U^2 + \Delta V^2 \geq 1 + |a.b|^2 - 2|a.b|\sqrt{(1 - \Delta U^2)(1 - \Delta V^2)}$$

• Abbott-Alzieu-Hall-Branciard tight state-independent uncertainty relations for qubits can be understood as UR with weak values [Hall-Pati-Wu, PRA (2016)].

Experimental test of UR (Peng Xue Group)



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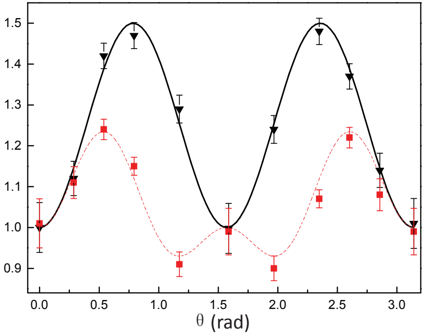

Figure : Experimental test of the uncertainty relation for two non-commuting unitary operators. The solid black line corresponds to the theoretical prediction of the LHS of the inequality, i.e., $\Delta U^2 + \Delta V^2$. The black triangles represent the sum of the measured uncertainties of ΔU^2 and ΔV^2 with the eleven states $|\psi_\theta\rangle$. The red dashed line corresponds to the theoretical prediction of the RHS of the inequality. The red squares represent the experimental results of the RHS of inequality with the eleven states $|\psi_\theta\rangle$.

Complementarity of weak values

Complementarity of weak values

- In quantum theory, complementarity imposes limitations on our ability to unambiguously define and measure aspects of quantum systems in a single measurement setup.
- Bohr "...it is only the mutual exclusion of any two experimental procedures, permitting the unambiguous definition of complementary physical quantities,...."
- In the context of weak measurements, it is possible that one can probe two complementary aspects of a quantum system at some price (e.g. introducing noise), as the apparatus interacts with the system weakly, allowing a gentle observation without disturbing the system too much.

- A strong type of complementarity between a given weak measurement procedure, and the mutually exclusive procedure obtained by interchanging the weak and the strong measurements.
- Weak measurements involving projection operators

$$A^a = |a\rangle\langle a|, \quad B^b = |b\rangle\langle b|,$$

corresponding to the eigenvalue decompositions of two nondegenerate observables $A = \sum_a aA^a$ and $B = \sum_b bB^b$.
- For a given initial state $|\psi\rangle$ there are then two complementary weak measurement scenarios: a weak measurement of projector A^a postselected on state $|b\rangle$, i.e, on $B = b$, and a weak measurement of projector B^b postselected on state $|a\rangle$, i.e., on $A = a$. The corresponding weak values

$$A^a_w(b|\psi) = \frac{\langle b|a\rangle\langle a|\psi\rangle}{\langle b|\psi\rangle},$$

$$B^b_w(a|\psi) = \frac{\langle a|b\rangle\langle b|\psi\rangle}{\langle a|\psi\rangle}.$$

- We see that the weak values connect wavefunctions directly in complementary bases. For example, we have

$$A_w^a(b|\psi)\,\psi(b) = \langle b|a\rangle\,\psi(a),$$
$$B_w^b(a|\psi)\,\psi(a) = \langle a|b\rangle\,\psi(b), \qquad (14)$$

where $\psi(a)$ and $\psi(b)$ are the wavefunctions in the eigenbasis representations of A and B, respectively.
- The interesting point to note here is $\psi(a)$ and $\psi(b)$ are directly related without a unitary transformation: the weak values act as 'filters' that connect two complementary aspects directly.

- Next, we ask can these two weak values be arbitrarily large at the same time? Strangely, not.
- First, note that the weak values for the projectors A^a and B^b can be expressed as the sum of the average of the projectors in the state $|\psi\rangle$ plus an anomalous part

$$A^a_w(b|\psi) = \langle A^a \rangle_\psi + \Delta_\psi A^a \frac{\langle b|\bar{\psi}_a\rangle}{\langle b|\psi\rangle},$$
$$B^b_w(a|\psi) = \langle B_b \rangle_\psi + \Delta_\psi B^b \frac{\langle a|\bar{\psi}_b\rangle}{\langle a|\psi\rangle}, \quad (15)$$

  where $\Delta_\psi A^a := (Var_\psi A^a)^{1/2}$ is the uncertainty of the projector in the state $|\psi\rangle$, $|\bar{\psi}_a\rangle$ is a state orthogonal to $|\psi\rangle$, and similar definitions hold for the other projector Π_b.
- This shows that the weak values of these projectors can be large, and lie outside the eigenvalue range $[0, 1]$ of the projectors.

- However, both weak values cannot be large at the same time.
- The product of these weak values satisfies

$$A_w^a(b|\psi)\, B_w^b(a|\psi) = |\langle a|b\rangle|^2 \leq 1. \tag{16}$$

- Thus, even though individually each of these weak values can be complex, with arbitrarily large moduluses, their product is real, independent of the pre-selected state, and bounded by unity.
- This represents a new kind of complementarity between the weak and strong components of quantum weak measurements.

- This type of complementarity also holds for the scenario of a weak momentum measurement postselected on the result of a strong position measurement, and its converse.
- In this case the corresponding weak values for state $|\psi\rangle$ are given by

$$P_w^p(x|\psi) = \frac{\langle x|p\rangle\langle p|\psi\rangle}{\langle x|\psi\rangle},$$

$$X_w^x(p|\psi) = \frac{\langle p|x\rangle\langle x|\psi\rangle}{\langle p|\psi\rangle},$$

and it is easily checked that the product of these two weak values satisfy the condition

$$X_w^x(p|\psi)\, P_w^p(x|\psi) = \frac{1}{2\pi\hbar}. \tag{17}$$

- Thus, the complementarity of weak values of two non-commuting projectors is a general feature of quantum systems, that holds in both finite and infinite dimensions.

Conclusions

- Relating average weak values to average quantum observables have many physical applications.
- Product representation formula can be used to recover quantum averages of products of observables from experimentally determined weak values of the observables.

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- Product representation formula can be used to recover quantum averages of products of observables from experimentally determined weak values of the observables.
- Heisenberg uncertainty relation is equivalent to a classical uncertainty relation for complex random variables
- Derive new preparation uncertainty relation for pairs of unitary operators.

Conclusions

- For two complementary weak measurement scenarios, in which the weakly and strongly measured observables are interchanged, there is a complementarity relation in the form of an upper bound on the product of the corresponding weak values.
- We hope that our results will open up new ways of thinking about uncertainty and complementarity relations using weak values.

THANK YOU!