Uncertainty and Entanglement

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Plan of the Talk

1. Introduction
   - Quantum Entanglement
   - Positive partial transpose
2. Detecting Entanglement
   - Witness through Angular Momentum addition
   - Uncertainty Relation Criterion

Quantum Entanglement

- Entanglement is a bizarre feature of Quantum Mechanics which allows non-locality in Quantum systems.
- Entanglement is regarded as a resource in quantum information processing. Many cryptographic tasks such as Quantum Key Distribution (QKD), Teleportation, Super-dense coding, Quantum Secret Sharing (QSS) has been achieved using this non-local correlation.
- Thus producing and detecting entanglement in lab is an important task. In the context of detecting entanglement, Bell inequality revolutionized the quantum world. Entanglement Witnesses (Horodecki et al. PLA1996) and uncertainty relation criterion (Nha PRL2007) are other ways of detecting entanglement.

Positive Partial Transpose

A bipartite quantum state is called separable if it can be written as convex combination of pure product states:

$$\sum_i p_i(\rho^A \otimes \rho^B)$$

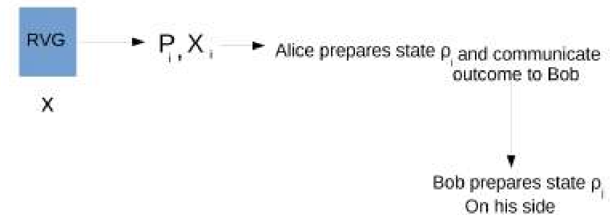
with $p_i \geq 0$ and $\sum_i p_i = 1$.

- Correlation contained in the bipartite quantum state

$$\sum_i p_i(\rho^A \otimes \rho^B)$$

is classical because above state can be created through local operation and classical communication (LOCC).

Separable states by LOCC



Positive Partial Transpose

- Transposition of a density matrix is a positive map, but it is not completely positive:

$$\rho^{A} \xrightarrow{Transposition} \rho^{A'} \geq 0$$

$$\rho^{AB} \xrightarrow{TranspositionB} \rho^{AB'} \ngeq 0$$

(In general)

- For separable bipartite states $\rho^{AB} = \sum_i p_i \rho^A \otimes \rho^B$ transposition with respect to one of the subsystem say B is $\rho^{AB'} = \sum_i p_i \rho^A \otimes \rho^{B^T}$ is again a positive matrix:

$$\rho^{AB'} \geq 0$$

Peres-Horodecki PPT Criterion

After doing the partial transposition, if one of the eigenvalue of the density matrix becomes negative, then for sure the state is entangled. Partial transposition criterion is sufficient condition for inseparability.

Entanglement witnesses

Definition

An observable W is called entanglement witness if and only if:
$Tr(W\rho) \geq 0$ for all separable ρ_s,
$Tr(W\rho) \leq 0$ for at least one entangled state ρ_e.
Then we know that the state ρ_e is entangled and we call state ρ_e is detected by witness W.

Examples:

- NPT Witnesses: Let us consider a NPT state ρ_e. Then ρ_e^{pt} has one negative eigenvalue λ and eigenvector $|\lambda\rangle$, then observable $|\lambda\rangle\langle\lambda|^{pt}$ is a witness detecting ρ_e, because
$Tr(|\lambda\rangle\langle\lambda|^{pt}\rho_e) = Tr(|\lambda\rangle\langle\lambda|\rho_e^{pt}) = \lambda \leq 0$ and
$Tr(|\lambda\rangle\langle\lambda|^{pt}\rho_s) = Tr(|\lambda\rangle\langle\lambda|\rho_s^{pt}) \geq 0$, because separable states are positive transpose.

Example 2

- A entanglement witness of the following form has been proposed in Sanpera et al. (PRA 2001):

$$W = \alpha\mathbb{I} - |\psi\rangle\langle\psi|$$

  Where $|\psi\rangle$ is an entangled state.
- The quantity $Tr(\rho|\psi\rangle\langle\psi|) = \langle\psi|\rho|\psi\rangle$ is called fidelity, which is the measure of closeness of $\{\rho, |\psi\rangle\langle\psi|\}$.
- Thus if fidelity exceeds a critical value α, then the state is entangled according to this criterion.

- The value of α should be such that

$$Tr(W\rho_s) \geq 0$$

for all separable states. α is proved to be square of the maximal Schimdt coefficient for $|\psi\rangle$ as proved by Bourennane et al. (PRL 2004).

- For two qubit maximally entangled state $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ above witness reads as:

$$W = \frac{1}{2} - |\psi\rangle \langle\psi|$$

Detecting Entanglement

Here, we will derive entanglement Witnesses based on physical principles:

- Angular Momentum Addition
- Uncertainty relation.

We found that uncertainty relation provides a stronger witness than bi-linear entanglement witnesses.

Detecting entanglement

- Relation

$$Tr(M\rho^{pt}) = Tr(M^{pt}\rho)$$

shifts partial transposition operation from ρ to operator M. In this way expectation value of observable M^{pt} is calculated w.r.t the bipartite density matrix ρ.

- In quantum mechanics expectation value of observables satisfy some constraints (two of them mentioned before). If a density matrix is separable, partial transpose of a density operator is again some density operator, hence these constraints on the expectation value of observables should still be satisfied.
- Thus violation of such constraints leads to entanglement. Since these constraints involve Hermitian observables, they can be tested experimentally.

Witness through Angular Momentum Addition

- Let us consider bipartite spin system with spin j and j' respectively. The eigenstates of the composite system can be characterized in the following two bases:
  $\vec{S}_1^2, \vec{S}_2^{\,2}, S_{1z}$ and S_{2z}
  $\vec{S}^2$, S_z, $\vec{S}_a^{\,2}$, $\vec{S}_b^{\,2}$.
  Where $\vec{S}^2 = (\vec{S}_a + \vec{S}_b)^2 = s(s+1)$ and $S_Z = S_{1z} + S_{2z}$
- Angular momentum addition allows, following values of s:

$$t, t-1, t-2, \ldots, t' \tag{1}$$

, where t= j+j' and t'=j-j'.

- Thus total angular momentum is bounded:

$$\vec{S}^2 \in [t'(t'+1), t(t+1)]$$

- Using relation,

$$\vec{S}^2 = (\vec{S_A} + \vec{S_B})^2 = S_A^2 + S_B^2 + 2\vec{S_A}.\vec{S_B}$$

  we can obtain bounds on

$$\vec{S_A}.\vec{S_B}$$

  as, $\vec{S_A}.\vec{S_B} \in [-j'(j+1), jj']$.

- Now we consider partial transpose,

$$\vec{S_A}.\vec{S_B} \to (\vec{S_A}.\vec{S_B})_{pt}$$

  .If the given state ρ is separable, $(\vec{S_A}.\vec{S_B})_{pt}$ must belongs to above given limit.

Condition for separability

$$< (\vec{S_A}.\vec{S_B})_{pt} > \in [-j'(j+1), jj']$$

- No separable state will violate above condition and violation will guaranty entanglement.

Two Qubit System

- For two qubit system following are the eigenstates:

s	m (eigenvalue of s_z)	Eigenstates
1	1	$\|00\rangle$
1	0	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$
1	-1	$\|11\rangle$
0	0	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$

- For the two qubit system, the condition of separability in terms of Pauli operator reads $\sigma(\vec{\sigma} = 2\vec{S})$,

$$G = < (\vec{\sigma_A} \cdot \vec{\sigma_B})_{pt} > = < \sigma_{x_A}\sigma_{x_B} - \sigma_{y_A}\sigma_{y_B} + \sigma_{z_A}\sigma_{z_B} >_\rho \in [-3, 1]$$

Where

$$(\sigma_x, \sigma_y, \sigma_z) \xrightarrow{Transposition} (\sigma_x, -\sigma_y, \sigma_z)$$

has been used.

- The upper bound of the condition provides following separability condition,

I - $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z \geq 0$

which detects the entangled state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and is similar to the bi-linear entanglement witness obtained from Schmidt number above. Thus above procedure generalizes these witness operators for two spin j and j$'$ *particle*.

- Since above is a condition for entanglement must be invariant under local unitary rotations, choosing local unitary $I \otimes \sigma_y$, we obtain:

$$\mathbb{I} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z \geq 0$$

which detects entangled state $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$.

Uncertainty Relation Criterion

- Every valid quantum state satisfies the Schrodinger-Robertson uncertainty (A stronger uncertainty than Heisenberg's) relation for the observables A and B having commutation relation $[A, B] = \imath C$,

$$\Delta(A)^2_\rho \Delta(B)^2_\rho \geq (\frac{1}{4} \mid< C >\mid^2 + \frac{1}{4} \mid< \{A, B\} > -2 < A >< B >\mid^2)_\rho. \tag{2}$$

- The partial transpose ρ^{pt} of a bipartite separable density matrix must be positive, which implies that it represents some physical quantum state, therefore must obey SRUR. Using, $Tr(\rho^{pt} M) = Tr(\rho M^{pt})$, for any observable M leads to,

Condition of separability

$$\Delta(A)^2_{pt} \Delta(B)^2_{pt} \geq \frac{1}{4} \mid< C^{pt} >\mid^2 + \frac{1}{4} \mid< \{A, B\}_{pt} > -2 < A^{pt} >< B^{pt} >\mid^2$$

.

Uncertainty relation Criterion

- Entanglement through uncertainty relation has been studied by R.Simon (PRL2000), Agarwal (JOPA2005) , Nha(PRL), Guhne(PRL2000), Gillet(PRA2009) and many others.
- The main problem of uncertainty relation criterion is to obtain useful operators A and B which detect entanglement.
- Agarwal et al. (JOPA2005) used uncertainty relation of SU(2) and SU(1,1) operator algebra to detect entanglement in non-Gaussian continuous variable states:
  SU(2) Algebra,

$$A = \frac{a^\dagger b + ab^\dagger}{2}, B = \frac{a^\dagger b - ab^\dagger}{2\imath}, C = \frac{a^\dagger a - b^\dagger b}{2}$$

.

SU(1,1) Algebra,

$$A = \frac{a^\dagger b^\dagger + ab}{2}, B = \frac{a^\dagger b^\dagger - ab}{2\imath}, C = \frac{a^\dagger a + b^\dagger b + 1}{2}$$

Only SU(1,1) proved to be useful in detecting non-Guassian states in optical modes.

- Uncertainty relation criterion can consolidate the bi-linear Schimdt witness if we use following operators:

$$A = \frac{1}{2}(\sigma_x \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_x)$$

$$B = \frac{1}{2}(\sigma_y \otimes \sigma_z - \sigma_z \otimes \sigma_y)$$

- Commutation and anti-communication relations of A and B are given by:
  $[A, B] = \imath C$ where
  $$C = \sigma_z \otimes \sigma_z + \sigma_y \otimes \sigma_y$$
  And, $\{A, B\} = 0$.
- Condition of separability obtained using above operators and uncertainty relation criterion,

Condition of separability

$$< D >^2 \geq \frac{1}{4} |< C^{pt} >|^2 + (< A^{pt} >^2 + < B^{pt} >^2) < D >$$

where

$$D = \frac{1}{2}(\mathbb{I} - \sigma_x \otimes \sigma_x).$$

and

$$C^{pt} = \sigma_z \otimes \sigma_z - \sigma_y \otimes \sigma_y$$

- The bi-linear witness can be obtained as special case of uncertainty relation criterion. Substituting $< A^{pt} >=< B^{pt} >= 0$ in above relation, we obtain

$$I - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z \geq 0,$$

.

- Since D is a positive operator, above provides stronger condition for entanglement than schimdt number entanglement witness.
- Above inequality detects Werner states $p \left|\psi\right\rangle \left\langle\psi\right| + (1-p)\mathbb{I}$ with[$p \geq \frac{1}{3}$] , which improves the limit of detection of other uncertainty relation criterion Gühne et al (PRL 2004)[$p \geq \frac{1}{\sqrt{3}}$], Gillet et al (PRA 2009)[$p \geq \frac{1}{2}$].

Summary and future directions

- Using partial transposition criterion on angular momentum addition of composite system and uncertainty relation robust entanglement witness has been obtained.
- Entanglement characterization in multi-partite density matrices is still an unsolved problem. We hope that above criteria will provide insight into this problem.

- Thank you for your patience.